

# INTERVAL ENFORCEABLE PROPERTIES OF FINITE GROUPS

WILLIAM DEMEO

**ABSTRACT.** We propose a classification of group properties according to whether they can be imposed on a group  $G$  by assuming the subgroup lattice of  $G$  contains an interval isomorphic to a given lattice. Suppose  $\mathcal{P}$  is a group property and there exists a lattice  $L$  such that if  $G$  is a group with  $L$  isomorphic to an interval  $[H, G]$  in  $\text{Sub}(G)$ , with  $H$  core-free, then  $G$  has property  $\mathcal{P}$ . We call such  $\mathcal{P}$  *core-free interval enforceable* (cf-IE). Among other things we show that if both a property and its negation could be proved cf-IE, this would solve an important problem in universal algebra.

**Key Words:** subgroup lattices, congruence lattices, group properties.

**2010 MSC:** Primary 20E15; Secondary 20D30, 20B10, 06B15, 08A30.

## 1. INTRODUCTION

The study of subgroup lattices has a long history, starting with Richard Dedekind [8] and Ada Rottlaender [28], and later a number of important contributions by Reinhold Baer, Øystein Ore, Michio Suzuki, Roland Schmidt, and many others (see Schmidt [29]). Much of this work focuses on the problem of inferring properties of a group  $G$  based on the structure of its lattice of subgroups,  $\text{Sub}(G)$ , or, conversely, inferring lattice theoretical properties of  $\text{Sub}(G)$  from properties of  $G$ . For some groups,  $\text{Sub}(G)$  determines  $G$  up to isomorphism. For example, this is true of the Klein-4 group, the alternating groups  $A_n$  ( $n \geq 4$ ), and every finite nonabelian simple group.<sup>1</sup> For other groups,  $\text{Sub}(G)$  is isomorphic to the subgroup lattices of all groups in an infinite class of nonisomorphic groups. Some examples are the following:  $\text{Sub}(G) \cong \mathbf{2}$  if and only if  $G$  is cyclic of prime order;  $\text{Sub}(G) \cong \mathbf{3}$  if and only if  $G$  is cyclic of order  $p^2$  for some prime  $p$ ;  $\text{Sub}(G) \cong \mathbf{2} \times \mathbf{2}$  if and only if  $G$  is cyclic of order  $pq$  for some primes  $p \neq q$ . At the other extreme, there are finite lattices which are not subgroup lattices.

In addition to results which infer properties of  $G$  from knowledge of the whole subgroup lattice  $\text{Sub}(G)$ , there are results which use only lattice theoretical properties of  $\text{Sub}(G)$  to obtain group theoretical properties of  $G$ .

---

*Date:* October 23, 2012.

<sup>1</sup>For a proof that every finite nonabelian simple group is determined by its subgroup lattice, see Theorem 7.8.1 of [29]. The proof relies on the CFSG Theorem.

For example, Øystein Ore [20, 21] proved that  $G$  is locally cyclic if and only if  $\text{Sub}(G)$  is distributive. Michio Suzuki [31] proved similar lattice theoretical characterizations for soluble and perfect groups. More recently, John Shareshian and Russ Woodroffe [30] prove that by comparing the lengths of certain maximal chains in  $\text{Sub}(G)$  one can determine whether or not  $G$  is soluble.

Historically, less attention was paid to the local structure of the subgroup lattice of a finite group, perhaps because it seemed that very little about  $G$  could be inferred from knowledge of, say, an upper interval  $[H, G] := \{K \mid H \leq K \leq G\}$  in the subgroup lattice of  $G$ . Recently, however, this topic has attracted more attention (e.g., [2, 5, 7, 16, 18, 22]), mostly owing to its connection to the most important open problem in universal algebra, the *finite lattice representation problem* (FLRP). This is the problem of characterizing the lattices that are (isomorphic to) congruence lattices of finite algebras (see, e.g., [6, 10, 23, 24]). There is a remarkable theorem relating this problem to intervals in subgroup lattices of finite groups.

**Theorem 1.1** (Pálffy and Pudlák [25]). *The following statements are equivalent:*

- (A) *Every finite lattice is isomorphic to the congruence lattice of a finite algebra.*
- (B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*

Let  $\mathcal{L}_0$  be the class of all finite lattices. For historical reasons, we take  $\mathcal{L}_3$  to be the class of those finite lattices that are isomorphic to congruence lattices of finite algebras, and we let  $\mathcal{L}_4$  be the class of lattices that are isomorphic to intervals in subgroup lattices of finite groups. It is not hard to see that  $\mathcal{L}_0 \supseteq \mathcal{L}_3 \supseteq \mathcal{L}_4$ . The theorem above states: if  $\mathcal{L}_0 = \mathcal{L}_3$  then  $\mathcal{L}_0 = \mathcal{L}_4$  (and conversely, of course). Note that the theorem does not say  $\mathcal{L}_3 = \mathcal{L}_4$ . It is possible that  $\mathcal{L}_0 \supsetneq \mathcal{L}_3 \supsetneq \mathcal{L}_4$ .

If the equivalent statements of the theorem above turn out to be true, and  $\mathcal{L}_0 = \mathcal{L}_3 = \mathcal{L}_4$ , we will say “the FLRP has a positive answer.” Otherwise, we say “the FLRP has a negative answer.” Thus, if we can find a finite lattice  $L$  for which it can be proved that there is no finite group  $G$  with  $L \cong [H, G]$  for some  $H < G$ , then it will follow from Theorem 1.1 that the FLRP has a negative answer.

In this paper we propose a new classification of group properties according to whether or not they can be imposed on a group  $G$  by assuming that  $\text{Sub}(G)$  has an upper interval isomorphic to some finite lattice. We believe this is a worthwhile endeavor on its own, but we also show how such classifications of group properties could be used to find a solution of the FLRP.

Suppose  $\mathcal{P}$  is a *group theoretical property* (defined in Section 2) and suppose there exists a finite lattice  $L$  such that if  $G$  is a finite group with  $L \cong [H, G]$  for some  $H \leq G$ , then  $G$  has property  $\mathcal{P}$ . We call such a  $\mathcal{P}$  an

*interval enforceable* (IE) property. An *interval enforceable class of groups* is a class of groups having property  $\mathcal{P}$ , for some IE property  $\mathcal{P}$ .

Merely requiring that  $L$  be isomorphic to an upper interval in  $\text{Sub}(G)$  seems quite weak, and it is difficult to find many properties that can be enforced by this assumption, so we strengthen the assumption and ask what properties can be imposed on a group  $G$  by some finite lattice  $L \cong [H, G]$  if we assume  $H$  is a *core-free* subgroup of  $G$ , that is,  $H$  contains no nontrivial normal subgroups of  $G$ . We call such properties *core-free interval enforceable* (cf-IE).

Extending this idea, we consider finite collections  $\mathcal{L}$  of finite lattices and ask what can be proved about a group  $G$  if one assumes that each  $L_i \in \mathcal{L}$  is isomorphic to an upper interval  $[H_i, G] \leq \text{Sub}(G)$ , with  $H_i$  core-free in  $G$ . Our main result (Theorem 3.6) connects this idea to the FLRP, as follows:

*Statement (B) of Theorem 1.1 is equivalent to each of the following statements:*

- (C) *For every finite lattice  $L$ , for every finite collection  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  of cf-IE classes of groups, there exists a finite group  $G \in \bigcap_{i=1}^n \mathfrak{X}_i$  such that  $L \cong [H, G]$  for some subgroup  $H$  that is core-free in  $G$ .*
- (D) *For every finite collection  $\mathcal{L}$  of finite lattices, there exists a finite group  $G$  such that for each  $L_i \in \mathcal{L}$  we have  $L_i \cong [H_i, G]$  for some subgroup  $H_i$  that is core-free in  $G$ .*

In fact, the arguments proving the equivalence of these statements are easily combined to show that the following is also equivalent to statement (B):

- (E) *For every finite collection  $\mathcal{L}$  of finite lattices, for every finite collection  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  of cf-IE classes of groups, there exists a finite group  $G \in \bigcap_{i=1}^n \mathfrak{X}_i$  such that for each  $L_i \in \mathcal{L}$  we have  $L_i \cong [H_i, G]$  for some subgroup  $H_i$  that is core-free in  $G$ .*

Core-free interval enforceable properties are intimately related to permutation representations of groups. If  $H$  is a core-free subgroup of  $G$ , then  $G$  has a faithful permutation representation  $\varphi : G \hookrightarrow \text{Sym}(G/H)$ . Let  $\langle G/H, \varphi(G) \rangle$  denote the algebra comprised of the right cosets  $G/H$  acted upon by right multiplication by elements of  $G$ ; that is,  $\varphi(g) : Hx \mapsto Hxg$ . It is well known that the congruence lattice of this algebra (i.e., the lattice of systems of imprimitivity) is isomorphic to the interval  $[H, G]$  in the subgroup lattice of  $G$ .<sup>2</sup> This puts statement (E) into perspective. If the FLRP has a positive answer, then no matter what we take as our finite collection  $\mathcal{L}$  – for example, we might take  $\mathcal{L}$  to be *all* finite lattices with at most  $N$  elements for some large  $N < \omega$  – we can always find a *single* finite group  $G$  such that every lattice in  $\mathcal{L}$  is a core-free upper interval in  $\text{Sub}(G)$ . As a

---

<sup>2</sup>See [19, Lemma 4.20] or [13, Theorem 1.5A].

result, the single finite group  $G$  must have so many faithful representations  $G \hookrightarrow \text{Sym}(G/H_i)$  with systems of imprimitivity isomorphic to  $L_i$ , one such representation for each distinct  $L_i \in \mathcal{L}$ . Moreover, the group  $G$  having this property can be chosen from the class  $\bigcap_{i=1}^n \mathfrak{X}_i$ , where  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  is an arbitrary collection of cf-IE classes of groups.

The main contribution of this work is the proposal of a new classification of group properties according to whether or not they can be enforced by the structure of a subgroup lattice interval. We classify a number of group properties in this way and prove, for example, that while insolubility is interval enforceable, solubility is not even core-free interval enforceable. Another contribution is Lemma 3.5, which generalizes a well known result as follows: If  $U$  and  $H$  are permuting subgroups of a group – that is  $UH = HU$  – then the set  $[U \cap H, U]^H := \{U \cap H \leq X \leq U \mid XH = HX\}$  is a sublattice of  $[U \cap H, U]$  that is isomorphic to the lattice  $[H, UH]$ . We have found this result to be very useful for proving that certain properties are interval enforceable.

It is well known that if  $U$  is normalized by  $H$  then the set of subgroups in  $[U \cap H, U]$  that are normalized by  $H$  is order isomorphic to the interval  $[H, UH]$  (see, e.g., [7]). In many situations, this result is quite useful for proving that certain properties are interval enforceable. However, in some applications (see, e.g., Sections 3.3 and 4 below) we can only assume that the groups  $H$  and  $U$  permute, but neither one normalizes the other. These instances requires the more general result mentioned above (and proved below as Lemma 3.5).

## 2. NOTATION AND DEFINITIONS

In this paper, *all groups and lattices are finite*. We use  $\mathfrak{G}$  to denote the class of all finite groups. Given a group  $G$ , we denote the set of subgroups of  $G$  by  $\text{Sub}(G)$ . The algebra  $\langle \text{Sub}(G), \wedge, \vee \rangle$  is a lattice where the  $\wedge$  (“meet”) and  $\vee$  (“join”) operations are defined for all  $H$  and  $K$  in  $\text{Sub}(G)$  by  $H \wedge K = H \cap K$  and  $H \vee K = \langle H, K \rangle =$  the smallest subgroup of  $G$  containing both  $H$  and  $K$ . We will refer to the set  $\text{Sub}(G)$  as a lattice, without explicitly mentioning the  $\wedge$  and  $\vee$  operations.

By  $H \leq G$  (resp.,  $H < G$ ) we mean  $H$  is a subgroup (resp., proper subgroup) of  $G$ . For  $H \leq G$ , the *core of  $H$  in  $G$* , denoted by  $\text{core}_G(H)$ , is the largest normal subgroup of  $G$  contained in  $H$ . If  $\text{core}_G(H) = 1$ , then we say that  $H$  is *core-free in  $G$* . For  $H \leq G$ , by the *interval  $[H, G]$*  we mean the set  $\{K \mid H \leq K \leq G\}$ , which is a sublattice of  $\text{Sub}(G)$ . That is,  $[H, G]$  is the lattice of those subgroups of  $G$  that contain  $H$ .<sup>3</sup> With this

---

<sup>3</sup>The reader may anticipate confusion arising from the conflict between our notation and the well-established notation for the *commutator subgroup*,  $[H, G] := \langle \{hgh^{-1}g^{-1} \mid h \in H, g \in G\} \rangle$ . However, we have found that context always makes clear which meaning is intended. In any case, we may refer to “the interval  $[H, G]$ ” or “the commutator  $[H, G]$ ” when extra clarity is required.

notation,  $\text{Sub}(G) = [1, G]$ . When viewing  $[H, G]$  as a sublattice of  $\text{Sub}(G)$ , we sometimes refer to it as an *upper interval*. Given an abstract lattice  $L$ , if there is no mention of specific groups  $H$  and  $G$ , then the expression  $L \cong [H, G]$  means “there exist (finite) groups  $H \leq G$  such that  $L$  is isomorphic to the interval  $\{K \mid H \leq K \leq G\}$  in the subgroup lattice of  $G$ .”

By a *group theoretical class*, or *class of groups*, we mean a collection  $\mathfrak{X}$  of groups that is closed under isomorphism: if  $G_0 \in \mathfrak{X}$  and  $G_1 \cong G_0$ , then  $G_1 \in \mathfrak{X}$ . A *group theoretical property*, or simply *property of groups*, is a property  $\mathcal{P}$  such that if a group  $G_0$  has property  $\mathcal{P}$  and  $G_1 \cong G_0$ , then  $G_1$  has property  $\mathcal{P}$ .<sup>4</sup> Thus if  $\mathfrak{X}_{\mathcal{P}}$  denotes the collection of groups with group theoretical property  $\mathcal{P}$ , then  $\mathfrak{X}_{\mathcal{P}}$  is a class of groups, and belonging to a class of groups is a group theoretical property. Therefore, we need not distinguish between a property of groups and the class of groups that possess this property. A group in the class  $\mathfrak{X}$  is called a  *$\mathfrak{X}$ -group*, and we sometimes write  $G \models \mathcal{P}$  to indicate that  $G$  has property  $\mathcal{P}$ .

If  $\mathcal{K}$  is a class of algebras (e.g., a class of groups), then we say that  $\mathcal{K}$  is *closed under homomorphic images* and we write  $\mathbf{H}(\mathcal{K}) = \mathcal{K}$  provided  $\varphi(G) \in \mathcal{K}$  whenever  $G \in \mathcal{K}$  and  $\varphi$  is a homomorphism of  $G$ . By the first isomorphism theorem for groups, this is equivalent to:  $G/N \in \mathcal{K}$  whenever  $G \in \mathcal{K}$  and  $N \trianglelefteq G$ . For algebras,  $\mathbf{H}(\mathcal{K}) = \mathcal{K}$  holds if and only if  $\mathbf{A}/\theta \in \mathcal{K}$  for all  $\mathbf{A} \in \mathcal{K}$  and all  $\theta \in \text{Con } \mathbf{A}$  (where  $\text{Con } \mathbf{A}$  denotes the lattice of congruence relations of  $\mathbf{A}$ ).

Apart from possible notational differences, the foregoing terminology is standard. We now introduce some new terminology that we find useful.<sup>5</sup> A group theoretical property  $\mathcal{P}$  (and the associated class  $\mathfrak{X}_{\mathcal{P}}$ ) is called

- *interval enforceable* (IE) provided

$$(\exists L) (\forall G) (L \cong [H, G] \longrightarrow G \models \mathcal{P})$$

- *core-free interval enforceable* (cf-IE) provided

$$(\exists L) (\forall G) ((L \cong [H, G] \bigwedge \text{core}_G(H) = 1) \longrightarrow G \models \mathcal{P})$$

- *minimal interval enforceable* (min-IE) provided there exists  $L$  such that if  $L \cong [H, G]$  for some group  $G$  of minimal order (with respect to  $L \cong [H, G]$ ), then  $G$  is a  $\mathcal{P}$ -group.

In this paper we will have little to say about min-IE properties. Nonetheless, we include this class in our list of new definitions because properties of this type arise often (see, e.g., [18]), and a primary aim of this paper is to formalize various notions of interval enforceability that we believe are useful in applications.

<sup>4</sup>It seems there is no single standard definition of *group theoretical class*. While some authors (e.g., [14], [4]) use the definition given here, others (e.g. [26], [27]) require that a group theoretical class contain groups of order 1.

<sup>5</sup>The author thanks Bjørn Kjos-Hanssen and David Ross for suggesting improvements to the wording of these definitions.

## 3. RESULTS

Clearly, if  $\mathcal{P}$  is an interval enforceable property, then it is also core-free interval enforceable. There is a simple sufficient condition under which the converse holds. First, if  $\mathcal{P}$  is a group property and  $\mathfrak{X}_{\mathcal{P}}$  the class of groups with property  $\mathcal{P}$ , then by  $\mathfrak{X}_{\mathcal{P}}^c$  we denote the class  $\{G \in \mathfrak{G} \mid G \not\in \mathcal{P}\}$  of groups that do not have property  $\mathcal{P}$ .

**Lemma 3.1.** *Suppose  $\mathcal{P}$  is a core-free interval enforceable property. If  $\mathbf{H}(\mathfrak{X}_{\mathcal{P}}^c) = \mathfrak{X}_{\mathcal{P}}^c$ , then  $\mathcal{P}$  is an interval enforceable property.*

*Proof.* Since  $\mathcal{P}$  is cf-IE there is a lattice  $L$  such that

$$(3.1) \quad (L \cong [H, G] \bigwedge \text{core}_G(H) = 1) \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}.$$

Under the assumption  $\mathbf{H}(\mathfrak{X}_{\mathcal{P}}^c) = \mathfrak{X}_{\mathcal{P}}^c$  we prove

$$(3.2) \quad L \cong [H, G] \longrightarrow G \in \mathfrak{X}_{\mathcal{P}}.$$

If (3.2) fails, then there is a group  $G \in \mathfrak{X}_{\mathcal{P}}^c$  with  $L \cong [H, G]$ . Let  $N = \text{core}_G(H)$ . Then  $L \cong [H/N, G/N]$  and  $H/N$  is core-free in  $G/N$  so, by hypothesis (3.1),  $G/N \in \mathfrak{X}_{\mathcal{P}}$ . But  $G/N \in \mathfrak{X}_{\mathcal{P}}^c$ , since  $\mathfrak{X}_{\mathcal{P}}^c$  is closed under homomorphic images.  $\square$

In [23], Péter Pálfi gives an example of a lattice that cannot occur as an upper interval in the subgroup lattice finite soluble group. (We give other examples in §3.3 and §4.) In his Ph.D. thesis [5], Alberto Basile proves that if  $G$  is an alternating or symmetric group, then there are certain lattices that cannot occur as upper intervals in  $\text{Sub}(G)$ . Another class of lattices with this property is described by Aschbacher and Shareshian in [2]. Thus, two classes of groups that are known to be at least cf-IE are the following:

- $\mathfrak{X}_0 = \mathfrak{S}^c = \text{insoluble finite groups};$
- $\mathfrak{X}_1 = \{G \in \mathfrak{G} \mid (\forall n < \omega) (G \neq A_n \wedge G \neq S_n)\},$

where  $A_n$  and  $S_n$  denote, respectively, the alternating and symmetric groups on  $n$  letters. Note that both classes  $\mathfrak{X}_0$  and  $\mathfrak{X}_1$  satisfy the hypothesis of 3.1. Explicitly,  $\mathfrak{X}_0^c = \mathfrak{S}$ , the class of soluble groups, is closed under homomorphic images, as is the class  $\mathfrak{X}_1^c$  of alternating and symmetric groups. Therefore, by Lemma 3.1,  $\mathfrak{X}_0$  and  $\mathfrak{X}_1$  are IE classes. By contrast, suppose there exists a finite lattice  $L$  such that

$$(L \cong [H, G] \bigwedge \text{core}_G(H) = 1) \longrightarrow G \text{ is subdirectly irreducible.}$$

Lemma 3.1 does not apply in this case since the class of subdirectly reducible groups is obviously not closed under homomorphic images.<sup>6</sup> In Sections 3.3 and 4 below we describe lattices with which we can prove that following classes are at least cf-IE:

---

<sup>6</sup>Recall, for groups *subdirectly irreducible* is equivalent to having a unique minimal normal subgroup. Every algebra, in particular every group  $G$ , has a subdirect decomposition into subdirectly irreducibles, say,  $G \hookrightarrow G/N_1 \times \cdots \times G/N_n$ , so there are always subdirectly irreducible homomorphic images.

- $\mathfrak{X}_2$  = the subdirectly irreducible groups;
- $\mathfrak{X}_3$  = the groups having no nontrivial abelian normal subgroups;
- $\mathfrak{X}_4 = \{G \in \mathfrak{G} \mid C_G(M) = 1 \text{ for all } 1 \neq M \trianglelefteq G\}$ .

We noted above that  $\mathfrak{X}_2$  fails to satisfy the hypothesis of 3.1. The same can be said of  $\mathfrak{X}_3$  and  $\mathfrak{X}_4$ . That is,  $\mathbf{H}(\mathfrak{X}_i^c) \neq \mathfrak{X}_i^c$  for  $i = 2, 3, 4$ . To verify this take  $H \in \mathfrak{X}_i$ ,  $K \in \mathfrak{X}_i^c$ , and consider  $H \times K$ . In each case ( $i = 2, 3, 4$ ) we see that  $H \times K$  belongs to  $\mathfrak{X}_i^c$ , but the homomorphic image  $(H \times K)/(1 \times K) \cong H$  does not.

**3.1. Negations of interval enforceable properties.** The following definition is useful: if a lattice  $L$  is isomorphic to an interval in the subgroup lattice of a finite group, then we call  $L$  *group representable*. Recall, Theorem 1.1 says that the FLRP has a negative answer if we can find a finite lattice that is not group representable.

Suppose there exists a property  $\mathcal{P}$  such that both  $\mathcal{P}$  and its negation  $\neg\mathcal{P}$  are interval enforceable by the lattices  $L$  and  $L_c$ , respectively. That is  $L \cong [H, G]$  implies  $G \in \mathfrak{X}_{\mathcal{P}}$  and  $L_c \cong [H_c, G_c]$  implies  $G_c \in \mathfrak{X}_{\mathcal{P}}^c$ . Then clearly the lattice in Figure 1 could not be group representable. As the next result

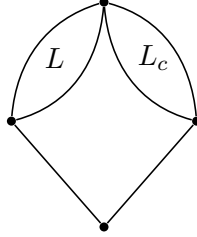


FIGURE 1.

shows, however, if a group property and its negation are interval enforceable by the lattices  $L$  and  $L_c$ , then already at least one of these lattices is not group representable.

**Lemma 3.2.** *If  $\mathcal{P}$  is a group property that is interval enforceable by a group representable lattice, then  $\neg\mathcal{P}$  is not interval enforceable by a group representable lattice.*

*Proof.* Assume the contrary. Then both  $\mathcal{P}$  and its negation  $\neg\mathcal{P}$  are interval enforceable by group representable lattices  $L$  and  $L_c$ , respectively. Let  $G$  and  $G_c$  be groups for which  $L \cong [H, G]$  and  $L_c \cong [H_c, G_c]$  for some  $H \leq G$  and  $H_c \leq G_c$ . Then the group  $G \times G_c$  has upper intervals  $L \cong [H \times G_c, G \times G_c]$  and  $L_c \cong [G \times H_c, G \times G_c]$ . Thus, by the interval enforceability assumptions, the group  $G \times G_c$  both is and is not a  $\mathfrak{X}_{\mathcal{P}}$ -group.  $\square$

To take a concrete example, insolubility is IE. However, solubility is obviously not IE. For, if  $L \cong [H, G]$  then for any insoluble group  $K$  we have  $L \cong [H \times K, G \times K]$ , and of course  $G \times K$  is insoluble. Note that here (and

in the proof of Lemma 3.2) the group  $H \times K$  at the bottom of the interval is not core-free. So a more interesting question is whether a property and its negation can both be cf-IE. Again, if such a property were found, a lattice of the form in Figure 1 would give a negative answer to the FLRP, though this requires additional justification to address the core-free aspect (see Section 3.3). We suspect the answer is no, as suggested by

**Conjecture 3.1.** If  $\mathcal{P}$  is core-free interval enforceable by a group representable lattice, then  $\neg\mathcal{P}$  is not core-free interval enforceable by a group representable lattice.

We confirm a special case of the foregoing conjecture – namely, the case when  $\mathcal{P}$  is insolubility. Indeed, the following lemma implies that the class of soluble groups, and more generally any class of groups that omits certain wreath products, cannot be core-free interval enforceable by a group representable lattice.

**Lemma 3.3.** *Suppose  $\mathcal{P}$  is core-free interval enforceable by a group representable lattice. Then, for any finite nonabelian simple group  $S$ , there exists a wreath product group of the form  $W = S \wr \bar{U}$  that has property  $\mathcal{P}$ .*

*Proof.* Let  $L$  be a group representable lattice such that if  $L \cong [H, G]$  and  $\text{core}_G(H) = 1$  then  $G \models \mathcal{P}$ . Since  $L$  is group representable, there exists a  $\mathcal{P}$ -group  $G$  with  $L \cong [H, G]$ . We apply an idea of Hans Kurzweil (see [17]) twice. Fix a finite nonabelian simple group  $S$ . Suppose the index of  $H$  in  $G$  is  $|G : H| = n$ . Then the action of  $G$  on the cosets of  $H$  induces an automorphism of the group  $S^n$  by permutation of coordinates. Denote this representation by  $\varphi : G \rightarrow \text{Aut}(S^n)$ , and let the image of  $G$  be  $\varphi(G) = \bar{G} \leq \text{Aut}(S^n)$ . The wreath product under this action is the group

$$U := S \wr_{\varphi} G = S^n \rtimes_{\varphi} G = S^n \rtimes \bar{G},$$

with multiplication given by

$$(s_1, \dots, s_n, x)(t_1, \dots, t_n, y) = (s_1 t_{x(1)}, \dots, s_n t_{x(n)}, xy),$$

for  $s_i, t_i \in S$  and  $x, y \in \bar{G}$ . (For the remainder of the proof, we suppress the semidirect product symbol and write, for example,  $S^n \bar{G}$  instead of  $S^n \rtimes \bar{G}$ .)

An illustration of the subgroup lattice of such a wreath product appears in Figure 2. Note that the interval  $[D, S^n]$ , where  $D$  denotes the diagonal subgroup of  $S^n$ , is isomorphic to  $\text{Eq}(n)'$ , the dual of the lattice of partitions of an  $n$ -element set. The dual lattice  $L'$  is an upper interval of  $\text{Sub}(U)$ , namely,  $L' \cong [D\bar{G}, U]$ .<sup>7</sup>

It is important to note (and we prove below) that if  $H$  is core-free in  $G$  – equivalently, if  $\ker \varphi = 1$  – then the foregoing construction results in the subgroup  $D\bar{G}$  being core-free in  $U$ . Therefore, by repeating the foregoing procedure, with  $H_1 = D\bar{G}$  denoting the (core-free) subgroup of  $U$  such that

---

<sup>7</sup>These facts, which were proved by Kurzweil in [17], are discussed in greater detail in [10, Section 2.2].



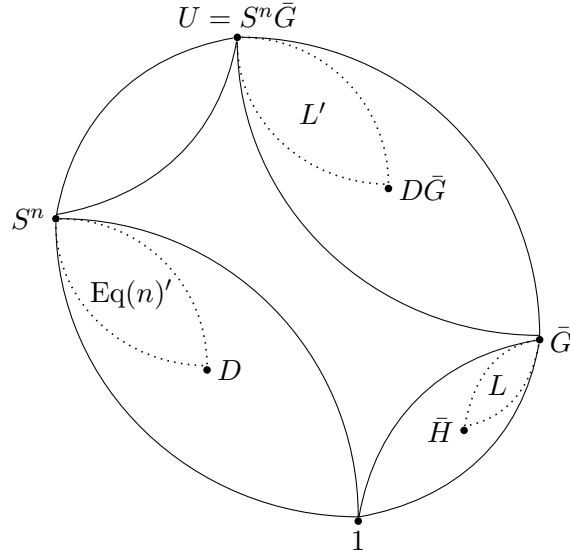


FIGURE 2. Hasse diagram illustrating some features of the subgroup lattice of the wreath product  $U$ .

$L' \cong [H_1, U]$ , we find that  $L = L'' \cong [D_1 \bar{U}, S^m \bar{U}]$ , where  $m = |U : H_1|$ , and  $D_1 \bar{U}$  denotes the diagonal subgroup of  $S^m$ . Since  $D_1 \bar{U}$  will be core-free in  $S^m \bar{U}$  then, it follows by the original hypothesis that  $S^m \bar{U} = S \wr \bar{U}$  must have property  $\mathcal{P}$ .

To complete the proof, we check that starting with a core-free subgroup  $H \leq G$  in the Kurzweil construction just described results in a core-free subgroup  $D\bar{G} \leq U$ . Let  $N = \text{core}_U(D\bar{G})$ . Then, for all  $w = (d, \dots, d, x) \in N$  and for all  $u = (t_1, \dots, t_n, g) \in U$ , we have  $uwu^{-1} \in N$ . Fix  $w = (d, \dots, d, x) \in N$ . We will choose  $u \in U$  so that the condition  $uwu^{-1} \in N$  implies  $x$  acts trivially on  $\{1, \dots, n\}$ . First note that if  $u = (t_1, \dots, t_n, 1)$ , then

$$\begin{aligned} unu^{-1} &= (t_1, \dots, t_n, 1)(d, \dots, d, x)(t_1^{-1}, \dots, t_n^{-1}, 1) \\ &= (t_1 d t_{x(1)}^{-1}, \dots, t_n d t_{x(n)}^{-1}, 1) \in N, \end{aligned}$$

and this implies that  $t_1 d t_{x(1)}^{-1} = t_2 d t_{x(2)}^{-1} = \dots = t_n d t_{x(n)}^{-1}$ . Suppose by way of contradiction that  $x(1) = j \neq 1$ . Then, since  $x$  is a permutation (hence, one-to-one),  $x(k) \neq j$  for each  $k \in \{2, 3, \dots, n\}$ . Pick one such  $k$  other than  $j$ . (This is possible since  $n = |G : H| > 2$ ; for otherwise  $H \trianglelefteq G$  contradicting  $\text{core}_G(H) = 1$ .) Since  $u \in U$  is arbitrary, we may assume  $t_1 = t_k$  and  $t_{x(1)} = t_j \neq t_{x(k)}$ . But this contradicts  $t_1 d t_{x(1)}^{-1} = t_k d t_{x(k)}^{-1}$ . Therefore,  $x(1) = 1$ . The same argument shows that  $x(i) = i$  for each  $1 \leq i \leq n$ , and we see that  $w = (d, \dots, d, x) \in N$  implies  $x \in \ker \varphi = 1$ .

This puts  $N$  below  $D$ , and the only normal subgroup of  $U$  that lies below  $D$  is the trivial group.  $\square$

By the foregoing result we conclude that a class of groups that does not include wreath products of the form  $S \wr G$ , where  $S$  is an arbitrary finite nonabelian simple group, is not a core-free interval enforceable class. The class of soluble groups is an example.

**3.2. Dedekind's rule and its consequences.** When  $A$  and  $B$  are subgroups of a group  $G$ , by  $AB$  we mean the set  $\{ab \mid a \in A, b \in B\}$ , and we write  $A \vee B$  or  $\langle A, B \rangle$  to denote the subgroup of  $G$  generated by  $A$  and  $B$ . Clearly  $AB \subseteq \langle A, B \rangle$ ; equality holds if and only if  $A$  and  $B$  *permute*, by which we mean  $AB = BA$ .

We will need the following standard theorem:<sup>8</sup>

**Theorem 3.4** (Dedekind's rule). *Let  $G$  be a group and let  $A, B$  and  $C$  be subgroups of  $G$  with  $A \leq B$ . Then,*

$$(3.3) \quad A(C \cap B) = AC \cap B, \quad \text{and}$$

$$(3.4) \quad (C \cap B)A = CA \cap B.$$

Our next lemma (Lemma 3.5) generalizes a standard result. We find this generalization useful for proving that certain properties are core-free interval enforceable. To state Lemma 3.5, we need some new notation. Let  $U$  and  $H$  be subgroups of a group, let  $U_0 = U \cap H$ , and consider the interval  $[U_0, U] = \{V \mid U_0 \leq V \leq U\}$ . It will be helpful to visualize part of the subgroup lattice of  $\langle U, H \rangle$ , as shown in Figure 3.

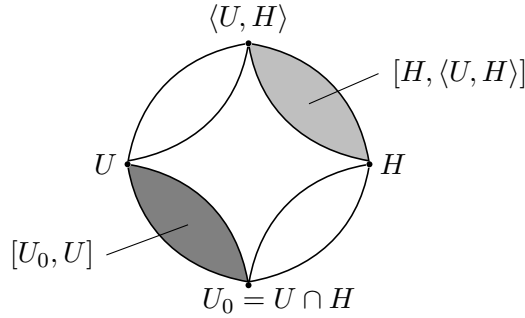


FIGURE 3. Some intervals in a subgroup lattice.

Recall that the usual isomorphism theorem for groups implies that if  $H$  is a normal subgroup of  $\langle U, H \rangle$ , then the interval  $[H, \langle U, H \rangle]$  is isomorphic to the interval  $[U \cap H, U]$ . The purpose of the next lemma is to relate these two intervals in cases where we drop the assumption  $H \trianglelefteq \langle U, H \rangle$  and add the assumption  $UH = \langle U, H \rangle$ .

<sup>8</sup>See, for example, page 122 of Rose, *A Course on Group Theory* [27].

If the two subgroups  $U$  and  $H$  permute, then we define

$$(3.5) \quad [U_0, U]^H = \{V \in [U_0, U] \mid VH = HV\},$$

which consists of those subgroups in  $[U_0, U]$  that permute with  $H$ .

If  $H$  normalizes  $U$  (which implies  $UH = HU$ ), then we define

$$(3.6) \quad [U_0, U]_H = \{V \in [U_0, U] \mid H \leq N_{UH}(V)\}.$$

This is the set consisting of those subgroups in  $[U_0, U]$  that are normalized by  $H$  (sometimes called *H-invariant subgroups* in  $[U_0, U]$ ). Notice that to even define  $[U_0, U]_H$  we must have  $H \leq N_{UH}(U)$ , and in this case, as we will see below, the sublattices  $[U_0, U]_H$  and  $[U_0, U]^H$  coincide.

We are now ready to state the main result relating the sets defined in (3.5) and (3.6) to the interval  $[H, UH]$ .

**Lemma 3.5.** *Suppose  $U$  and  $H$  are permuting subgroups of a group. Let  $U_0 = U \cap H$ . Then*

- (i)  $[H, UH] \cong [U_0, U]^H \leq [U_0, U]$ .
- (ii) If  $U \trianglelefteq UH$ , then  $[U_0, U]_H = [U_0, U]^H \leq [U_0, U]$ .
- (iii) If  $H \trianglelefteq UH$ , then  $[U_0, U]_H = [U_0, U]^H = [U_0, U]$ .

*Remarks.* Part (ii) of Lemma 3.5 is a standard result (see, e.g., [7]). It seems likely that part (i) of the lemma is also well known, though we have not seen it elsewhere. Since  $G = UH$  is a group, the hypothesis of (ii) is equivalent to  $H \leq N_G(U)$ , and the hypothesis of (iii) is equivalent to  $U \leq N_G(H)$ . Part (i) of the lemma says that when two subgroups permute, we can identify the interval above either one of them with the sublattice of subgroups below the other that permute with the first. Part (ii) is similar except we identify the interval above  $H$  with the sublattice of  $H$ -invariant subgroups below  $U$ . Once we have proved (i), the proof of (iii) follows trivially from the standard isomorphism theorem for groups, so we omit the details.

*Proof.* To prove (i), we first show that the following maps are inverse order isomorphisms:

$$(3.7) \quad \begin{aligned} \varphi : [H, UH] \ni X &\mapsto U \cap X \in [U_0, U]^H \\ \psi : [U_0, U]^H \ni V &\mapsto VH \in [H, UH]. \end{aligned}$$

Then we show that  $[U_0, U]^H$  is a sublattice of  $[U_0, U]$ , that is,  $[U_0, U]^H \leq [U_0, U]$ .

Fix  $X \in [H, UH]$ . We claim that  $U \cap X \in [U_0, U]^H$ . Indeed,

$$(U \cap X)H = UH \cap X = HU \cap X = H(U \cap X).$$

The first equality holds by (3.4) since  $H \leq X$ , the second holds by assumption, and the third by (3.3). This proves  $U \cap X \in [U_0, U]^H$ . Moreover, by the first equality,  $\psi \circ \varphi(X) = (U \cap X)H = UH \cap X = X$ , so  $\psi \circ \varphi$  is the identity on  $[H, UH]$ .

If  $V \in [U_0, U]^H$ , then  $VH = HV$  implies  $VH \in [H, UH]$ . Also,  $\varphi \circ \psi$  is the identity on  $[U_0, U]^H$ , since  $\varphi \circ \psi(V) = VH \cap U = V(H \cap U) = VU_0 = V$ ,

by (3.3). This proves that  $\varphi$  and  $\psi$  are inverses of each other on the sets indicated, and it's easy to see that they are order preserving:  $X \leq Y$  implies  $U \cap X \leq U \cap Y$ , and  $V \leq W$  implies  $VH \leq WH$ . Therefore,  $\varphi$  and  $\psi$  are inverse order isomorphisms.

To complete the proof of (i), we show that  $[U_0, U]^H$  is a sublattice of  $[U_0, U]$ . Suppose  $V_1$  and  $V_2$  are subgroups in  $[U_0, U]$  that permute with  $H$ . It is easy to see that their join  $V_1 \vee V_2 = \langle V_1, V_2 \rangle$  also permutes with  $H$ , so we just check that their intersection permutes with  $H$ . Fix  $x \in V_1 \cap V_2$  and  $h \in H$ . We show  $xh = h'x'$  for some  $h' \in H$ ,  $x' \in V_1 \cap V_2$ . Since  $V_1$  and  $V_2$  permute with  $H$ , we have  $xh = h_1v_1$  and  $xh = h_2v_2$  for some  $h_1, h_2 \in H$ ,  $v_1 \in V_1$ ,  $v_2 \in V_2$ . Therefore,  $h_1v_1 = h_2v_2$ , which implies  $v_1 = h_1^{-1}h_2v_2 \in HV_2$ , so  $v_1$  belongs to  $V_1 \cap HV_2$ . Note that  $V_1 \cap HV_2$  is below both  $V_1$  and  $U \cap HV_2 = \varphi\psi(V_2) = V_2$ . Therefore,  $v_1 \in V_1 \cap HV_2 \leq V_1 \cap V_2$ , and we have proved that  $xh = h_1v_1$  for  $h_1 \in H$  and  $v_1 \in V_1 \cap V_2$ , as desired.

To prove (ii), assuming  $U \trianglelefteq G := UH$ , we show that if  $U_0 \leq V \leq U$ , then  $VH = HV$  if and only if  $H \leq N_G(V)$ . If  $H \leq N_G(V)$ , then  $VH = HV$  (even when  $U \not\trianglelefteq G$ ). Suppose  $VH = HV$ . We must show  $(\forall v \in V)(\forall h \in H) hvh^{-1} \in V$ . Fix  $v \in V$ ,  $h \in H$ . Then,  $hv = v'h'$  for some  $v' \in V$ ,  $h' \in H$ , since  $VH = HV$ . Therefore,  $v'h'h^{-1} = hvh^{-1} = u$  for some  $u \in U$ , since  $H \leq N_G(U)$ . This proves that  $hvh^{-1} \in VH \cap U = V(H \cap U) = VU_0 = V$ , as desired.  $\square$

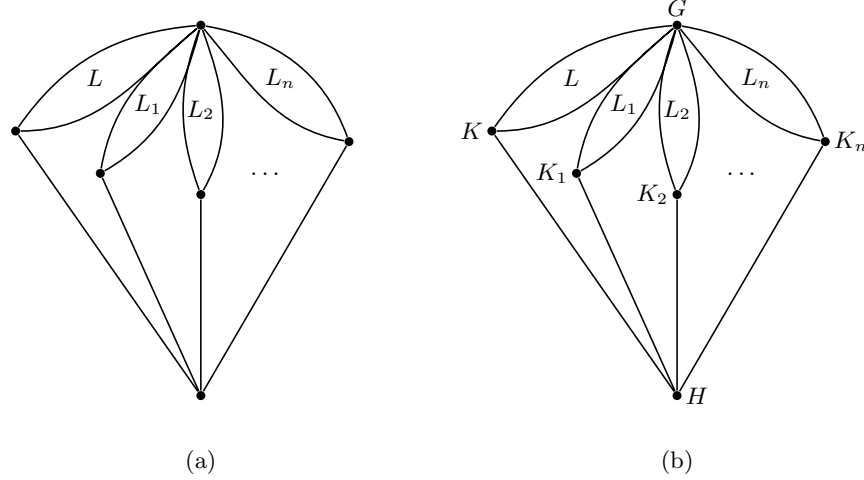
**3.3. Parachute lattices.** We now prove the equivalence of statements (B), (C), and (D) mentioned in Section 1. (That statement (E) of Section 1 is also equivalent to (B) follows easily from the arguments given below, so we omit the details.)

**Theorem 3.6.** *The following statements are equivalent:*

- (B) *Every finite lattice is isomorphic to an interval in the subgroup lattice of a finite group.*
- (C) *For every finite lattice  $L$ , for every finite collection  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  of cf-IE classes of groups, there exists a finite group  $G \in \bigcap_{i=1}^n \mathfrak{X}_i$  such that  $L \cong [H, G]$  for some core-free subgroup  $H \leq G$ .*
- (D) *For every finite collection  $\mathcal{L}$  of finite lattices, there exists a finite group  $G$  such that each  $L_i \in \mathcal{L}$  is isomorphic to  $[H_i, G]$  for some core-free subgroup  $H_i \leq G$ .*
- (E) *For every finite collection  $\mathcal{L}$  of finite lattices, for every finite collection  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  of cf-IE classes of groups, there exists a finite group  $G \in \bigcap_{i=1}^n \mathfrak{X}_i$  such that for each  $L_i \in \mathcal{L}$  we have  $L_i \cong [H_i, G]$  for some subgroup  $H_i$  that is core-free in  $G$ .*

*Remark.* By (C), the FLRP would have a negative answer if we could find a collection  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  of cf-IE classes such that  $\bigcap_{i=1}^n \mathfrak{X}_i$  is empty.

FIGURE 4. The parachute construction.



*Proof.* We prove the equivalence of (B) and (C). Obviously (C) implies (B). Assume (B) holds and let  $L$  be any finite lattice. Suppose  $\mathfrak{X}_1, \dots, \mathfrak{X}_n$  is a collection of cf-IE enforceable classes of groups. Then there exist finite lattices  $L_1, \dots, L_n$  such that  $L_i \cong [H_i, G_i]$  with  $\text{core}_G(H_i) = 1$  implies  $G_i \in \mathfrak{X}_i$ . Construct a new lattice, denoted  $\mathcal{P} = \mathcal{P}(L, L_1, \dots, L_n)$ , as shown in the Hasse diagram of Figure 4 (a). Note that the bottoms of the  $L_i$  sublattices are atoms in  $\mathcal{P}$ . By (B), there exist groups  $H < G$  with  $\mathcal{P} \cong [H, G]$ . We can assume  $H$  is a core-free subgroup of  $G$ . (If not, replace  $G$  and  $H$  with  $G/N$  and  $H/N$ , where  $N = \text{core}_G(H)$ , and note that  $\mathcal{P} \cong [H, G] \cong [H/N, G/N]$ .) Let  $K, K_1, \dots, K_n$  be the subgroups of  $G$  in which  $H$  is maximal and for which  $L \cong [K, G]$  and  $L_i \cong [K_i, G]$ ,  $1 \leq i \leq n$  (Figure 4 (b)). If we can prove for each  $1 \leq i \leq n$  that  $\text{core}_G(K_i) = 1$ , then  $G \in \mathfrak{X}_i$  by hypothesis, and it will follow that  $G \in \bigcap_{i=1}^n \mathfrak{X}_i$ , proving that (B) implies (C).

If  $L \cong \mathbf{2}$ , the two element lattice, or if  $L_j \cong \mathbf{2}$  for all  $1 \leq j \leq n$ , then the theorem is vacuously true. So we can assume without loss of generality that  $L \not\cong \mathbf{2}$  and that there is at least one  $1 \leq j \leq n$  for which  $L_j \not\cong \mathbf{2}$ . Let  $N$  be a minimal normal subgroup of  $G$  and suppose  $N \leq K_i$ . Since  $H$  is core-free,  $K_i = NH$ . Suppose  $K_i \neq K$ . Note that the groups  $K$  and  $K_i$  permute:

$$KK_i = KNH = NKH = NHK = K_iK.$$

Therefore, by Lemma 3.5 (i) we see that the interval  $[K, G] \cong L$  must be isomorphic to a sublattice of the interval  $[H, K_i] \cong \mathbf{2}$ , but this contradicts  $L \not\cong \mathbf{2}$ . Suppose instead that  $K = K_i$ . Note that  $K \neq K_j$ , and since  $K = NH$ , we see that  $K$  and  $K_j$  permute. Therefore, by Lemma 3.5 (i) again, the lattice  $L_j \not\cong \mathbf{2}$  is isomorphic to a sublattice of  $[H, K] \cong \mathbf{2}$ , which is impossible. This proves that  $NH = G$  for all  $N \trianglelefteq G$ , so each  $K_i$  is core-free.

The proof that statements (B) and (D) are equivalent follows by a similar construction. Roughly, if  $\mathcal{L} = \{L_1, \dots, L_n\}$ , we form the lattice  $\mathcal{P} = \mathcal{P}(L_1, \dots, L_n)$ . If (B) holds, then there exists a group  $G$  with  $\mathcal{P} \cong [H, G]$  and  $\text{core}_G(H) = 1$ . The proof that each  $K_i$  is core-free, where  $L_i \cong [K_i, G]$ , is similar to the argument above.  $\square$

By a *parachute lattice*, denoted  $\mathcal{P}(L_1, \dots, L_m)$ , we mean a lattice just like the one illustrated in Figure 4 (a), but with the lattices  $L_1, \dots, L_m$  appearing as the upper intervals.

Next we prove that any group that has a nontrivial parachute lattice as an upper interval in its subgroup lattice must have some rather special properties.

**Lemma 3.7.** *Let  $\mathcal{P} = \mathcal{P}(L_1, \dots, L_n)$  with  $n \geq 2$  and  $|L_i| > 2$  for all  $i$ , and suppose  $\mathcal{P} \cong [H, G]$ , with  $H$  core-free in  $G$ .*

- (i) *If  $1 \neq N \trianglelefteq G$ , then  $NH = G$  and  $C_G(N) = 1$ .*
- (ii)  *$G$  is subdirectly irreducible and insoluble.*

*Remark.* If a subgroup  $N \leq G$  is abelian, then  $N \leq C_G(N)$ , so (i) implies that every nontrivial normal subgroup of  $G$  is nonabelian.

*Proof.* (i) Let  $1 \neq N \trianglelefteq G$ . Then  $N \not\leq H$ , since  $H$  is core-free in  $G$ . Therefore,  $H < NH$ . As in Section 3.3, we let  $K_i$  denote the subgroups of  $G$  corresponding to the atoms of  $\mathcal{P}$ . Then  $H$  is covered by each  $K_i$ , so  $K_j \leq NH$  for some  $1 \leq j \leq n$ . Suppose, by way of contradiction, that  $NH < G$ . By assumption,  $n \geq 2$  and  $|L_i| > 2$ . Thus for any  $i \neq j$  we have  $K_i \leq Y < Z < G$  for some subgroups  $Y$  and  $Z$  which satisfy  $(NH) \cap Z = H$  and  $(NH) \vee Y = G$ . Also,  $(NH)Y = NY$  is a group, so  $(NH)Y = NH \vee Y = G$ . But then, by Dedekind's rule, we have

$$Y = HY = ((NH) \cap Z)Y = (NH)Y \cap Z = G \cap Z = Z,$$

contrary to  $Y < Z$ . This contradiction proves that  $NH = G$ .

To prove that  $C_G(N) = 1$ , let  $M$  be a minimal normal subgroup of  $G$  contained in  $N$ . It suffices to prove  $C_G(M) = 1$ . Assume the contrary. Then, since  $C_G(M) \trianglelefteq N_G(M) = G$ , it follows by what we just proved that  $C_G(M)H = G$ . Consider any  $H < K < G$ . Then  $1 < M \cap K < M$  (strictly, by Lemma 3.5). Now  $M \cap K$  is normalized by  $H$  and centralized (hence normalized) by  $C_G(M)$ . Therefore,  $M \cap K \trianglelefteq C_G(M)H = G$ , contradicting the minimality of  $M$ .

To prove (ii) we first show that  $G$  has a unique minimal normal subgroup. Let  $M$  be a minimal normal subgroup of  $G$  and let  $N \trianglelefteq G$  be any normal subgroup not containing  $M$ . We show that  $N = 1$ . Since both subgroups are normal, the commutator subgroup  $[M, N]$  lies in the intersection  $M \cap N$ , which is trivial by the minimality of  $M$ . Thus,  $M$  and  $N$  centralize each other. In particular,  $N \leq C_G(M) = 1$ , by (i).

Finally, since  $G$  has a unique minimal normal subgroup that is nonabelian (see the remark preceding the proof),  $G$  is insoluble.  $\square$

To summarize what we have thus far, the lemmas above imply that (B) holds if and only if every finite lattice is an interval  $[H, G]$ , with  $H$  core-free in  $G$ , where

- (i)  $G$  is insoluble, not alternating, and not symmetric;
- (ii)  $G$  has a unique minimal normal subgroup  $M$  which satisfies  $MH = G$  and  $C_G(M) = 1$ ; in particular,  $M$  is nonabelian and  $\text{core}_G(X) = 1$  for all  $H \leq X < G$ .

Finally, we recall that the structure of the unique minimal normal subgroup can be described as follows (see, e.g., [13, Theorem 4.3.A]):

- (iii)  $M = T_0 \times \cdots \times T_{r-1}$ , where  $T_i$  are simple minimal normal subgroups of  $M$  which are conjugate (under conjugation by elements of  $G$ ). Thus,  $M$  is a direct power of a simple group  $T$ .

In fact, when  $C_G(M) = 1$ , as in our application, we can specify these conjugates more precisely. Let  $T$  be any minimal normal subgroup of  $M$ . Note that  $T$  is simple. Let  $N = N_H(T) = \{h \in H : T^h = T\}$  be the normalizer of  $T$  in  $H$ . Then the proof of the following lemma is routine, so we omit it.

**Lemma 3.8.** *If  $H/N = \{N, h_1N, \dots, h_{k-1}N\}$  is a full set of left cosets of  $N$  in  $H$ , then  $k = r$  and  $M = T_0 \times \cdots \times T_{r-1} = T \times T^{h_1} \times \cdots \times T^{h_{r-1}}$ .*

We conclude this section by noting that other researchers, such as Baddeley, Börner, and Lucchini, have proved similar results for a more general class of lattices called *LP-lattices*.<sup>9</sup> These authors observe that a group having an L-P lattice as an upper interval in its subgroup lattice must be a *quasiprimitive permutation group*. We remark that a parachute lattice in which each panel  $L_i$  has  $|L_i| > 2$  is an LP-lattice, so Lemma 3.7 follows from theorems of Baddeley, Börner, Lucchini, et al. (cf. [3], [7]). However, the main purpose of our construction is not only to provide a quick route to Lemma 3.7, but also to demonstrate a natural way to insert arbitrary finite lattices  $L_i$  as upper intervals  $[K_i, G]$  in  $\text{Sub}(G)$ , with  $K_i$  core-free in  $G$ , so that once we establish a number of cf-IE properties, it will follow that *every* finite lattice  $L$  must be isomorphic to an upper interval  $L \cong [K, G]$  for some  $G$  satisfying all of these cf-IE properties, if the FLRP is to have a positive answer.

We conclude this section by formalizing the remarks of the previous sentence. Given two group theoretical properties  $\mathcal{P}_1, \mathcal{P}_2$ , we write  $\mathcal{P}_1 \rightarrow \mathcal{P}_2$  to denote that property  $\mathcal{P}_1$  implies property  $\mathcal{P}_2$ . In other words,  $G \models \mathcal{P}_1$  only if  $G \models \mathcal{P}_2$ . Thus  $\rightarrow$  provides a natural partial order on any given set of properties, as follows:

$$\mathcal{P}_1 \leq \mathcal{P}_2 \iff \mathcal{P}_1 \rightarrow \mathcal{P}_2 \iff \mathfrak{X}_{\mathcal{P}_1} \subseteq \mathfrak{X}_{\mathcal{P}_2},$$

where  $\mathfrak{X}_{\mathcal{P}_i} = \{G \in \mathfrak{G} \mid G \models \mathcal{P}_i\}$ . The following is an immediate corollary of the parachute construction described above.

---

<sup>9</sup>An LP-lattice is one in which every element except 0 and 1 is a non-modular element.

**Corollary 3.9.** *If  $\mathcal{P} = \{\mathcal{P}_i \mid i \in \mathcal{I}\}$  is a collection of (cf-)IE properties, then  $\bigwedge \mathcal{P}$  is (cf-)IE.*

The conjunction  $\bigwedge \mathcal{P}$  corresponds to the class  $\bigcap_{i \in \mathcal{I}} \mathfrak{X}_{\mathcal{P}_i} = \{G \in \mathfrak{G} \mid (\forall i \in \mathcal{I}) G \models \mathcal{P}_i\}$ .

#### 4. AN APPLICATION

We consider an application that demonstrates the utility of Lemma 3.5 for identifying certain core-free interval enforceable properties. The lattice that we study in this section has special relevance to the finite lattice representation problem (FLRP).

In prior work,<sup>10</sup> we considered whether, for every lattice  $L$  with at most  $n$  elements, there exists a finite algebra with a congruence lattice that is isomorphic to  $L$ . For  $n = 6$ , the problem had already been solved. In fact, Aschbacher [1] and Watatani [32] prove that every lattice with at most 6 elements is group representable (as defined in Section 3.1 above). For  $n = 7$ , although we have not found them all as intervals in subgroup lattices, we have found congruence lattice representations for all lattice with at most 7 elements with one exception (see [11], [10]). The exceptional lattice, which we call  $L_7$ , appears in Figure 5. Thus  $L_7$  is the smallest lattice for which we have not found a representation of the form  $L_7 \cong \text{Con } \mathbf{A}$  for some finite algebra  $\mathbf{A}$ .

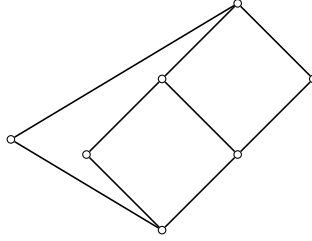


FIGURE 5. The exceptional seven element lattice,  $L_7$ .

Suppose  $\mathbf{A}$  is a finite algebra with  $\text{Con } \mathbf{A} \cong L_7$ , and suppose  $\mathbf{A}$  is of minimal cardinality among those algebras having a congruence lattice isomorphic to  $L_7$ . Then  $\mathbf{A}$  must be isomorphic to a transitive  $G$ -set. (This fact is proved in the forthcoming article [12].) Therefore, if  $L_7$  is representable, we can assume there is a finite group  $G$  with a core-free subgroup  $H < G$  such that  $L_7$  is isomorphic to the interval sublattice  $[H, G] \leq \text{Sub}(G)$ . In this section we present some restrictions on the possible groups for which this can occur.

<sup>10</sup>Universal algebra and lattice theory seminar, University of Hawaii, 2010-11; participants: Ralph Freese, Tristan Holmes, Peter Jipsen, Bill Lampe, J.B. Nation and the author.



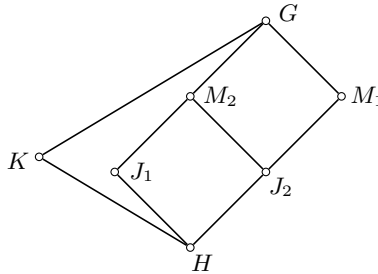
The first restriction, which is the easiest to observe, is that  $G$  must act primitively on the cosets of one of its maximal subgroups. This suggests the possibility of describing  $G$  in terms of the Aschbacher-O’Nan-Scott Theorem which characterizes primitive permutation groups.<sup>11</sup> Ultimately, the goal would be to find enough restrictions on  $G$  so as to rule out all finite groups. As yet, we have not achieved this goal. However, the new results in this section reduce the possibilities to special subclasses of the Aschbacher-O’Nan-Scott Theorem. This paves the way for future studies to focus on these subclasses when searching for a group representation of  $L_7$ , or proving that none exists.

**Proposition 4.1.** *Suppose  $G$  is a finite group with  $[H, G] \cong L_7$  for some core-free subgroup  $H < G$ . Then the following hold.*

- (i)  $G$  is a primitive permutation group.
- (ii) If  $N \triangleleft G$ , then  $C_G(N) = 1$ .
- (iii)  $G$  contains no non-trivial abelian normal subgroup.
- (iv)  $G$  is insoluble.
- (v)  $G$  is subdirectly irreducible.
- (vi) With the possible exception of at most one maximal subgroup, all proper subgroups in the interval  $[H, G]$  are core-free.

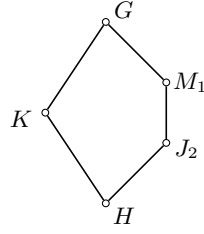
*Remark.* It is obvious that (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv), and (ii)  $\Rightarrow$  (v), but we include these easy consequences in the statement of the result for emphasis; for, although the hard work will be in proving (ii) and (vi), our main goal is the pair of restrictions (iii) and (v), which allow us to rule out a number of the O’Nan-Scott types describing primitive permutation groups.

Assume the hypotheses of the proposition above. In particular, throughout this section *all groups are finite,  $H$  is a core-free subgroup of  $G$ , and  $[H, G] \cong L_7$* . Label the seven subgroups of  $G$  in the interval  $[H, G]$  as in the following diagram:



We now prove the foregoing proposition through a series of claims. The first thing to notice about the interval  $[H, G]$  is that  $K$  is a *non-modular element* of the interval. This means that there is a pentagonal ( $N_5$ ) sublattice of the interval with  $K$  as the incomparable proper element. (See the diagram below, for example.)

<sup>11</sup>The author thanks John Shareshian for this suggestion. See the comments at MathOverflow [9].



Using this non-modularity property of  $K$ , it is easy to prove the following

**Claim 4.1.**  $K$  is a core-free subgroup of  $G$ .

*Proof.* Let  $N = \text{core}_G(K)$ . If  $N \leq X$  for some  $X \in \{M_1, M_2, J_1, J_2\}$ , then  $N < X \cap K = H$ , so  $N = 1$  (since  $H$  is core-free). If  $N \not\leq X$  for all  $X \in \{M_1, M_2, J_1, J_2\}$ , then  $NJ_2 = G$ . But then Dedekind's rule leads to the following contradiction:

$$J_2 \leq M_1 \implies J_2 = J_2(N \cap M_1) = J_2N \cap M_1 = G \cap M_1 = M_1.$$

Therefore,  $N = 1$ . □

Note that (i) of the proposition follows from Claim 4.1. Since  $K$  is core-free,  $G$  acts faithfully on the cosets  $G/K$  by right multiplication. Since  $K$  is a maximal subgroup, the action is primitive.

The next claim is slightly harder than the previous one as it requires the more general consequence of Dedekind's rule that we established above in Lemma 3.5 (i).

**Claim 4.2.**  $J_1$  and  $J_2$  are core-free subgroups of  $G$ .

*Proof.* First note that if  $N \trianglelefteq G$  then the subgroup  $NH$  permutes<sup>12</sup> with any subgroup containing  $H$ . To see this, let  $H \leq X \leq G$  and note that

$$NHX = NX = XN = XHN = XNH,$$

since  $H \leq X$  and  $N \trianglelefteq G$ .

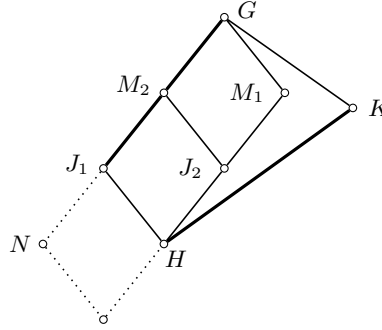
Suppose  $1 \neq N \leq J_1$  for some  $N \triangleleft G$ . Then  $NH = J_1$ , so  $J_1$  and  $K$  are permuting subgroups. Since  $J_1K = G$  and  $J_1 \cap K = H$ , Lemma 3.5 yields

$$[J_1, G] \cong [H, K]^{J_1} := \{X \in [H, K] \mid J_1X = XJ_1\}.$$

But this is impossible since  $[H, K]^{J_1} \leq [H, K] \cong \mathbf{2}$ , while  $[J_1, G] \cong \mathbf{3}$ . This proves that  $\text{core}_G(J_1) = 1$ . The intervals involved in the argument are drawn with bold lines in the following diagram.

---

<sup>12</sup>Recall, for subgroups  $X$  and  $Y$  of a group  $G$ , we define the sets  $XY = \{xy \mid x \in X, y \in Y\}$ , and  $YX = \{yx \mid x \in X, y \in Y\}$ , and we say that  $X$  and  $Y$  are *permuting subgroups* (or that  $X$  and  $Y$  permute, or that  $X$  permutes with  $Y$ ) provided the two sets  $XY$  and  $YX$  coincide, in which case the set forms a group:  $XY = \langle X, Y \rangle = YX$ .



The proof that  $J_2$  is core-free is similar. Suppose  $1 \neq N \leq J_2$  where  $N \triangleleft G$ . Then  $NH = J_2$  and the subgroups  $J_2$  and  $K$  permute. Therefore,  $[H, K]^{J_2} \cong [J_2, G]$ , by Lemma 3.5, which is a contradiction since  $[H, K]^{J_2} \leq [H, K] \cong \mathbf{2}$ , while  $[J_2, G] \cong \mathbf{2} \times \mathbf{2}$ .  $\square$

Now that we know  $K, J_1, J_2$  are each core-free in  $G$ , we use this information to prove that at least one of the other maximal subgroups,  $M_1$  or  $M_2$ , is core-free in  $G$ , thereby establishing (vi) of the proposition. We will also see that  $G$  is subdirectly irreducible, proving (v). The proof of (ii) will then follow from the same argument used to prove Lemma 3.5 (ii), which we repeat below.

**Claim 4.3.** Either  $M_1$  or  $M_2$  is core-free in  $G$ . If  $M_2$  has non-trivial core and  $N \triangleleft G$  is contained in  $M_2$ , then  $C_G(N) = 1$  and  $G$  is subdirectly irreducible.

*Proof.* Suppose  $M_2$  has non-trivial core. Then there is a minimal normal subgroup  $1 \neq N \triangleleft G$  contained in  $M_2$ . Since  $H, J_1, J_2$  are core-free,  $NH = M_2$ . Consider the centralizer,  $C_G(N)$ , of  $N$  in  $G$ . Of course, this is a normal subgroup of  $G$ .<sup>13</sup> If  $C_G(N) = 1$ , then, since minimal normal subgroups centralize each other,  $N$  must be the unique minimal normal subgroup of  $G$ . Furthermore,  $M_1$  must be core-free in this case. Otherwise  $N \leq M_1 \cap M_2 = J_2$ , contradicting  $\text{core}_G(J_2) = 1$ . Therefore, in case  $C_G(N) = 1$  we conclude that  $G$  is subdirectly irreducible and  $M_1$  is core-free.

We now prove that the alternative,  $C_G(N) \neq 1$ , does not occur. This case is a bit more challenging and must be split up into further subcases, each of which leads to a contradiction. Throughout, the assumption  $1 \neq N \leq M_2$  is in force, and it helps to keep in mind the diagram in Figure 6.

Suppose  $C_G(N) \neq 1$ . Then, since  $C_G(N) \trianglelefteq G$ , and since  $H, J_1, J_2, K$  are core-free, it's clear that  $C_G(N)H \in \{G, M_1, M_2\}$ . We consider each case separately.

<sup>13</sup>The centralizer of a normal subgroup  $N \trianglelefteq G$  is itself normal in  $G$ . For, it is the kernel of the conjugation action of  $G$  on  $N$ . Thus,  $C_G(N) \trianglelefteq N_G(N) = G$ .

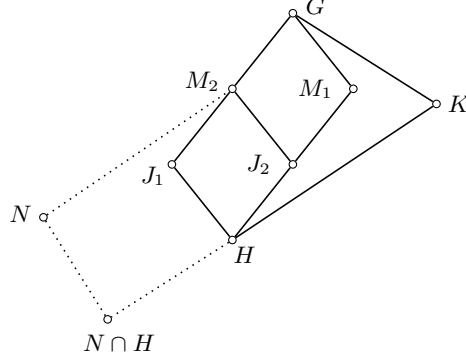


FIGURE 6. Hasse diagram illustrating the cases in which  $M_2$  has non-trivial core:  $1 \neq N \leq M_2$  for some  $N \triangleleft G$ .

- Case 1:* Suppose  $C_G(N)H = G$ . Note that  $N \cap H < N \cap J_1 < N$  (strictly). The subgroup  $N \cap J_1$  is normalized by  $J_1$  and by  $C_G(N)$ , and so it is normal in  $C_G(N)J_1 \geq C_G(N)H = G$ , contradicting the minimality of  $N$ . Thus, the case  $C_G(N)H = G$  does not occur.
- Case 2:* Suppose  $C_G(N)H = M_1$ . The subgroup  $N \cap J_1$  is normalized by both  $H$  and  $C_G(N)$ . For,  $C_G(N)$  centralizes, hence normalizes, every subgroup of  $N$ . Therefore,  $N \cap J_1$  is normalized by  $C_G(N)H = M_1$ . Of course, it's also normalized by  $J_1$ , so  $N \cap J_1$  is normalized by the set  $M_1J_1$ , so it's normalized by the group generated by that set, which is  $\langle M_1, J_1 \rangle = G$ .<sup>14</sup> The conclusion is that  $N \cap J_1 \triangleleft G$ . Since  $J_1$  is core-free,  $N \cap J_1 = 1$ . But this contradicts the (by now familiar) consequence of Dedekind's rule:

$$H < J_1 < M_2 \implies N \cap H < N \cap J_1 < N \cap M_2.$$

Therefore,  $C_G(N)H = M_1$  does not occur.

- Case 3:* Suppose  $C_G(N)H = M_2$ . The subgroup  $N \cap M_1$  is normalized by both  $H$  and  $C_G(N)$ . Therefore,  $N \cap M_1$  is normalized by  $C_G(N)H = M_2$ . Of course, it's also normalized by  $M_1$ , so  $N \cap M_1$  is normalized by  $\langle M_1, M_2 \rangle = G$ . The conclusion is that  $N \cap M_1 \triangleleft G$ . By minimality of the normal subgroup  $N$ , we must have either  $N \cap M_1 = 1$  or  $N \cap M_1 = N$ . The former equality implies  $N \cap J_2 = 1$ , which contradicts the strict inequalities of Dedekind's rule,

$$(4.1) \quad H < J_2 < M_2 \implies N \cap H < N \cap J_2 < N \cap M_2,$$

while the latter equality ( $N \cap M_1 = N$ ) implies that  $N \leq M_1 \cap M_2 = J_2$  which contradicts  $\text{core}_G(J_2) = 1$ .

□

<sup>14</sup>Actually, the set is already a group in this case since  $M_1J_1 = C_G(N)HJ_1 = J_1C_G(N)H = J_1M_1$ .

We have proved that either  $M_1$  or  $M_2$  is core-free in  $G$ , and we have shown that, if  $M_2$  has non-trivial core, then  $G$  is subdirectly irreducible. In fact, we proved that  $C_G(N) = 1$  for the unique minimal normal subgroup  $N$  in this case. It remains to prove that  $G$  is subdirectly irreducible in case  $M_1$  has non-trivial core. The argument is similar to the foregoing, and we omit some of the details that can be checked exactly as above.

**Claim 4.4.** If  $M_1$  has non-trivial core and  $N \triangleleft G$  is contained in  $M_1$ , then  $C_G(N) = 1$  and  $G$  is subdirectly irreducible.

*Proof.* If  $M_1$  has non-trivial core, then there is a minimal normal subgroup  $N \triangleleft G$  contained in  $M_1$ . We proved above that  $M_2$  must be core-free in this case, so either  $C_G(N)H = G$ ,  $C_G(N)H = M_1$ , or  $C_G(N) = 1$ . The first case is easily ruled out exactly as in Case 1 above. The second case is handled by the argument we used in Case 3. Indeed, if we suppose  $C_G(N)H = M_1$ , then  $N \cap M_2$  is normalized by both  $H$  and  $C_G(N)$ , hence by  $M_1$ . It is also normalized by  $M_2$ , so  $N \cap M_2 \triangleleft G$ . Thus, by minimality of  $N$ , and since  $M_2$  is core-free,  $N \cap M_2 = 1$ . But then  $N \cap J_2 = 1$ , leading to a contradiction similar to (4.1) but with  $M_1$  replacing  $M_2$ . Therefore, the case  $C_G(N)H = M_1$  does not occur, and we have proved  $C_G(N) = 1$ .  $\square$

So far we have proved that all intermediate proper subgroups in the interval  $[H, G]$  are core-free except possibly at most one of  $M_1$  or  $M_2$ . Moreover, we proved that if one of the maximal subgroups has non-trivial core, then there is a unique minimal normal subgroup  $N \triangleleft G$  with trivial centralizer,  $C_G(N) = 1$ . As explained above,  $G$  is subdirectly irreducible in this case, since minimal normal subgroups centralize each other.

In order to prove (ii), there remains only one case left to check, and the argument is by now very familiar.

**Claim 4.5.** If each  $H \leq X < G$  is core-free and  $N$  is a minimal normal subgroup of  $G$ , then  $C_G(N) = 1$ .

*Proof.* Let  $N$  be a minimal normal subgroup of  $G$ . Then, by the core-free hypothesis we have  $NH = G$ . Fix a subgroup  $H < X < G$ . Then  $N \cap H < N \cap X < N$ . The subgroup  $N \cap X$  is normalized by  $H$  and by  $C_G(N)$ . If  $C_G(N) \neq 1$ , then  $C_G(N)H = G$ , by the core-free hypothesis, so  $N \cap X \triangleleft G$ , contradicting the minimality of  $N$ . Therefore,  $C_G(N) = 1$ .  $\square$

Finally, we note that the claims above taken together prove (ii), and thereby complete the proof of the proposition. For if  $G$  is subdirectly irreducible with unique minimal normal subgroup  $N$ , and if  $C_G(N) = 1$ , then all normal subgroups (which necessarily lie above  $N$ ) must have trivial centralizers.

We conclude with a final observation which helps us describe the O’Nan-Scott type of a group that has  $L_7$  as an interval in its subgroup lattice. By what we have proved,  $G$  acts primitively on the cosets of  $K$ , and it also acts primitively on the cosets of at least one of  $M_1$  or  $M_2$ . Suppose

$M_1$  is core-free so that  $G$  is a primitive permutation group in its action on cosets of  $M_1$ , and let  $N$  be a minimal normal subgroup of  $G$ . As we have seen,  $N$  has trivial centralizer, so it is nonabelian and is the unique minimal normal subgroup of  $G$ . Now, we have seen that  $NH \geq M_2$  in this case, so  $H < J_2 < NH$  implies that  $N \cap M_1 \neq 1$ . Similarly, if we had started out by assuming that  $M_2$  is core-free, then  $NH \geq M_1$ , and  $H < J_2 < NH$  would imply that  $N \cap M_2 \neq 1$ .

By an elementary result (see [15, Lemma 8.5]), if  $G$  acts transitively on a set  $\Omega$  with stabilizer  $G_\omega$ , then a subgroup  $N \leq G$  acts transitively on  $\Omega$  if and only if  $NG_\omega = G$ , and  $N$  is *regular*<sup>15</sup> if and only if in addition  $N \cap G_\omega = 1$ . Thus, in the present application, we see that the action of  $N$  on the cosets of the core-free maximal subgroup  $M_i$  is not regular. Consequently,  $G$  is characterized by the Aschbacher-O’Nan-Scott Theorem as being either almost simple, of product action type, or of diagonal type (see [13, Theorem 4.6A]).

## REFERENCES

- [1] Aschbacher, M.: On intervals in subgroup lattices of finite groups. J. Amer. Math. Soc. **21**(3), 809–830 (2008). URL <http://dx.doi.org/10.1090/S0894-0347-08-00602-4>
- [2] Aschbacher, M., Shalev, J.: Restrictions on the structure of subgroup lattices of finite alternating and symmetric groups. J. Algebra **322**(7), 2449–2463 (2009). URL <http://dx.doi.org/10.1016/j.jalgebra.2009.05.042>
- [3] Baddeley, R., Lucchini, A.: On representing finite lattices as intervals in subgroup lattices of finite groups. J. Algebra **196**(1), 1–100 (1997). URL <http://dx.doi.org/10.1006/jabr.1997.7069>
- [4] Ballester-Bolínches, A., Ezquerro, L.M.: Classes of finite groups, *Mathematics and Its Applications (Springer)*, vol. 584. Springer, Dordrecht (2006)
- [5] Basile, A.: Second maximal subgroups of the finite alternating and symmetric groups. Ph.D. thesis, Australian National University, Canberra (2001)
- [6] Berman, J.: Congruence lattices of finite universal algebras. Ph.D. thesis, University of Washington (1970). URL <http://db.tt/mXUVtzSr>
- [7] Börner, F.: A remark on the finite lattice representation problem. In: Contributions to general algebra, 11 (Olomouc/Velké Karlovice, 1998), pp. 5–38. Heyn, Klagenfurt (1999)
- [8] Dedekind, R.: Über die Anzahl der Ideal-classes in den verschiedenen Ordnungen eines endlichen Körpers. In: Festschrift zur Saecularfeier des Geburtstages von C. F. Gauss, pp. 1–55. Vieweg, Braunschweig (1877). See Ges. Werke, Band I, 1930, 105–157
- [9] DeMeo, W.: Given a lattice  $L$  with  $n$  elements, are there finite groups  $H < G$  such that  $L \cong$  the lattice of subgroups between  $H$  and  $G$ ? MathOverflow. URL <http://mathoverflow.net/questions/85724>. (accessed 2012-05-08)
- [10] DeMeo, W.: Congruence lattices of finite algebras. Ph.D. thesis, University of Hawai’i at Mānoa, Honolulu, HI (2012). URL <http://arxiv.org/abs/1204.4305>

---

<sup>15</sup>Recall, a transitive permutation group  $N$  is *acts regularly* on a set  $\Omega$  provided the stabilizer subgroup of  $N$  is trivial. Equivalently, every non-identity element of  $N$  is fixed-point-free. Equivalently,  $N$  is regular on  $\Omega$  if and only if for each  $\omega_1, \omega_2 \in \Omega$  there is a unique  $n \in N$  such that  $n\omega_1 = \omega_2$ . In particular,  $|N| = |\Omega|$ .

- [11] DeMeo, W.: Expansions of finite algebras and their congruence lattices (2012). URL <http://arxiv.org/abs/1205.1106>. To appear in *Algebra Universalis*
- [12] DeMeo, W., Freese, R.: Congruence lattices of intransitive G-sets (2012). *Preprint*
- [13] Dixon, J.D., Mortimer, B.: Permutation groups, *Graduate Texts in Mathematics*, vol. 163. Springer-Verlag, New York (1996)
- [14] Doerk, K., Hawkes, T.: Finite soluble groups, *de Gruyter Expositions in Mathematics*, vol. 4. Walter de Gruyter & Co., Berlin (1992)
- [15] Isaacs, I.M.: Finite group theory, *Graduate Studies in Mathematics*, vol. 92. American Mathematical Society, Providence, RI (2008)
- [16] Köhler, P.:  $M_7$  as an interval in a subgroup lattice. *Algebra Universalis* **17**(3), 263–266 (1983). URL <http://dx.doi.org/10.1007/BF01194535>
- [17] Kurzweil, H.: Endliche Gruppen mit vielen Untergruppen. *J. Reine Angew. Math.* **356**, 140–160 (1985). URL <http://dx.doi.org/10.1515/crll.1985.356.140>
- [18] Lucchini, A.: Intervals in subgroup lattices of finite groups. *Comm. Algebra* **22**(2), 529–549 (1994). URL <http://dx.doi.org/10.1080/00927879408824862>
- [19] McKenzie, R.N., McNulty, G.F., Taylor, W.F.: Algebras, lattices, varieties. Vol. I. Wadsworth & Brooks/Cole, Monterey, CA (1987)
- [20] Ore, Ø.: Structures and group theory. I. *Duke Math. J.* **3**(2), 149–174 (1937). DOI 10.1215/S0012-7094-37-00311-9
- [21] Ore, Ø.: Structures and group theory. II. *Duke Math. J.* **4**(2), 247–269 (1938). DOI 10.1215/S0012-7094-38-00419-3
- [22] Pálffy, P.P.: On Feit’s examples of intervals in subgroup lattices. *J. Algebra* **116**(2), 471–479 (1988). URL [http://dx.doi.org/10.1016/0021-8693\(88\)90230-X](http://dx.doi.org/10.1016/0021-8693(88)90230-X)
- [23] Pálffy, P.P.: Intervals in subgroup lattices of finite groups. In: Groups ’93 Galway/St. Andrews, Vol. 2, *London Math. Soc. Lecture Note Ser.*, vol. 212, pp. 482–494. Cambridge Univ. Press, Cambridge (1995). URL <http://dx.doi.org/10.1017/CB09780511629297.014>
- [24] Pálffy, P.P.: Groups and lattices. In: Groups St. Andrews 2001 in Oxford. Vol. II, *London Math. Soc. Lecture Note Ser.*, vol. 305, pp. 428–454. Cambridge Univ. Press, Cambridge (2003). URL <http://dx.doi.org/10.1017/CB09780511542787.014>
- [25] Pálffy, P.P., Pudlák, P.: Congruence lattices of finite algebras and intervals in subgroup lattices of finite groups. *Algebra Universalis* **11**(1), 22–27 (1980). URL <http://dx.doi.org/10.1007/BF02483080>
- [26] Robinson, D.J.S.: A course in the theory of groups, *Graduate Texts in Mathematics*, vol. 80, second edn. Springer-Verlag, New York (1996)
- [27] Rose, J.S.: A course on group theory. Dover Publications Inc., New York (1994). Reprint of the 1978 original [Cambridge University Press; MR0498810 (58 #16847)]
- [28] Rottlaender, A.: Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen. *Math. Z.* **28**(1), 641–653 (1928). URL <http://dx.doi.org/10.1007/BF01181188>
- [29] Schmidt, R.: Subgroup lattices of groups, *de Gruyter Expositions in Mathematics*, vol. 14. Walter de Gruyter & Co., Berlin (1994)
- [30] Shreshian, J., Woodroffe, R.: A new subgroup lattice characterization of finite solvable groups. *Journal of Algebra* **351**, 448–458 (2012). DOI 10.1016/j.jalgebra.2011.10.032
- [31] Suzuki, M.: On the lattice of subgroups of finite groups. *Trans. Amer. Math. Soc.* **70**, 345–371 (1951)
- [32] Watatani, Y.: Lattices of intermediate subfactors. *J. Funct. Anal.* **140**(2), 312–334 (1996). URL <http://dx.doi.org/10.1006/jfan.1996.0110>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA, SC 29208, UNITED STATES